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STABLE MANIFOLDS IN THE METHOD OF AVERAGING

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ABSTRACT. Consider the differential equation $\dot{z} = \varepsilon f(z,t,\varepsilon)$, where f is T-periodic in t and $\varepsilon > 0$ is a small parameter, and the averaged equation $\dot{z} = \overline{f}(z) := (1/T)\int_0^T f(z,t,0)\,dt$. Suppose the averaged equation has a hyperbolic equilibrium at $z = 0$ with stable manifold $\overline{W}$. Let $\beta_\varepsilon(t)$ denote the hyperbolic T-periodic solution of $\dot{z} = \varepsilon f(z,t,\varepsilon)$ near $z \equiv 0$. We prove a result about smooth convergence of the stable manifold of $\beta_\varepsilon(t)$ to $\overline{W} \times \mathbf{R}$ as $\varepsilon \to 0$. The proof uses ideas of Vanderbauwhede and van Gils about contractions on a scale of Banach spaces.

1. Introduction. Consider

$$\dot{z} = \varepsilon f(z,t,\varepsilon), \tag{1}$$

where $f\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^1 and T-periodic in t. Consider also the averaged equation

$$\dot{z} = \overline{f}(z) := \frac{1}{T}\int_0^T f(z,t,0)\,dt. \tag{2}$$

Suppose $\overline{f}(0) = 0$ and $D\overline{f}(0)$ is hyperbolic, i.e., has no eigenvalues on the imaginary axis. Then it is well known that for each small $\varepsilon > 0$ there is a hyperbolic T-periodic solution $\beta_\varepsilon(t)$ of (1) near $z \equiv 0$. Our goal is to study the dependence of the stable manifold of $\beta_\varepsilon(t)$ on ε as ε approaches 0.

Let E_- (resp. E_+) denote the stable (resp. unstable) subspace of $D\overline{f}(0)$. Let B_δ denote the ball of radius δ about 0 in $\mathbf{R}^n$. A result of Hale [**3**, pp. 166–167] states that there are constants $k > 0$, $\delta > 0$, and $\varepsilon_0 > 0$ such that the following is true: for each ε with $0 < \varepsilon < \varepsilon_0$, there is a Lipschitz continuous function $(E_- \cap B_\delta) \times \mathbf{R} \to E_+$ with Lipschitz constant k whose graph is a local stable manifold of $\beta_\varepsilon(t)$. (Note that the stable manifold of $\beta_\varepsilon(t)$ is a subset of $\mathbf{R}^n \times \mathbf{R}$, where the second variable is time.) This result, which is proved using only a Lipschitz assumption on f, gives a "uniform" description of the stable manifolds of $\beta_\varepsilon(t)$ for small $\varepsilon > 0$, but does not provide information about their limit as ε approaches 0.

We now state our result:

THEOREM 1.1. *Let $r \geq 1$ and assume f is C^s where $s \geq 3(r+1)$. Then there exists $\delta > 0$ and a C^r function $\psi\colon (E_- \cap B_\delta) \times \mathbf{R} \times [0,\delta) \to E_+$ such that* (1) *for $\varepsilon > 0$, the graph of $\psi(\cdot,\cdot,\varepsilon)$ is a local stable manifold of $\beta_\varepsilon(t)$;* (2) *there is a local stable manifold $\overline{W} \subset \mathbf{R}^n$ of 0 for $z = \overline{f}(z)$ such that the graph of $\psi(\cdot,\cdot,0)$ is $\overline{W} \times \mathbf{R}$.*

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Of course, an analogous result holds for unstable manifolds. We remark that one knows from the usual stable manifold theorem that the restriction of ψ to $(E_- \cap B_\delta) \times \mathbf{R} \times (0, \delta)$ is C^s. We make no claim that this result is the best possible; we hope that the differentiability assumption can eventually be reduced to $s \geq r+1$.

Let us indicate an application of Theorem 1.1. Suppose $\dot{z} = \overline{f}(z)$ has an orbit homoclinic to the hyperbolic equilibrium at $z = 0$. One wishes to use Melnikov's method [**3**] to determine how this homoclinic orbit breaks as ε increases from 0. Melnikov's method involves the computation of an integral that is supposed to represent the derivative with respect to ε at $\varepsilon = 0$ of the separation between the stable and unstable manifolds of $\beta_\varepsilon(t)$. One way to justify Melnikov's method requires knowing that this derivative in fact exists; this is a consequence of Theorem 1.1. We remark that the fact that this derivative exists also follows from a version of Hartman's Theorem for equations like (1) that is due to Murdock and Robinson [**6**]. A consequence of their theorem is that there is an asymptotic expansion in power of ε for the position of the stable or unstable manifold of $\beta_\varepsilon(t)$.

2. Outline of proof. We consider equation (1), where $f\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is at least C^1 and is T-periodic in t. We consider also the averaged equation (2). We assume $\overline{f}(0) = 0$ and $A := D\overline{f}(0)$ is hyperbolic. We first make a change of coordinates that takes the T-periodic solutions of (1) to zero.

For small $\varepsilon \neq 0$, let $\beta(t, \varepsilon) := \beta_\varepsilon(t)$ be the T-periodic solution of (1) near $z \equiv 0$, and let $\beta(t, 0) := 0$.

PROPOSITION 2.1. *In the above situation, assume f is C^s, $s \geq 1$. Then there exists $\varepsilon_1 > 0$ such that $\beta(t, \varepsilon)$ is C^s on $\mathbf{R} \times (-\varepsilon_1, \varepsilon_1)$. If we make the T-periodic coordinate change $z = y + \beta(t, \varepsilon)$, then* (1) *becomes*

$$\dot{y} = \varepsilon g(y, t, \varepsilon), \tag{3}$$

where g is C^s on $\mathbf{R}^n \times \mathbf{R} \times (-\varepsilon_1, \varepsilon_1)$ and T-periodic in t. Moreover, $g(0, t, \varepsilon) \equiv 0$ and $\overline{g}(y) = \overline{f}(y)$, where

$$\overline{g}(y) := \frac{1}{T} \int_0^T g(y, t, 0)\, dt.$$

Thus (3) has the solution $y \equiv 0$ for each ε. Proposition 2.1 is proved in §3.

Next we use averaging to make (3) autonomous to order $2r + 1$:

PROPOSITION 2.2. *Let $r \geq 1$ and assume f is C^s, $s \geq 3(r+1)$. Let g be as in Proposition 2.1. Then there is a C^{r+3} change of coordinates $y = \varphi(x, t, \varepsilon)$, with $\varphi(x, t, 0) = x$ and $\varphi(0, t, \varepsilon) \equiv 0$, that transforms* (3) *to*

$$\dot{x} = \varepsilon A x + \sum_{i=0}^{2r} \varepsilon^{i+1} g_i(x) + \varepsilon^{2r+2} h(x, t, \varepsilon). \tag{4}$$

The coordinate change is defined on a set of the form $B_\rho \times \mathbf{R} \times (-\rho, \rho)$ in $xt\varepsilon$-space. Moreover, g_i and h are at least C^{r+2}, h is T-periodic in t, $g_i(0) = 0$, $h(0, t, \varepsilon) \equiv 0$, $Dg_0(0) = 0$, and $Ax + g_0(x) = \overline{g}(x) = \overline{f}(x)$.

The proof of Proposition 2.2 is in §4.

We now scale time by $\tau := \varepsilon t$, which converts (4) to

$$\frac{dx}{d\tau} = Ax + \sum_{i=0}^{2r} \varepsilon^i g_i(x) + \varepsilon^{2r+1} h\left(x, \frac{\tau}{\varepsilon}, \varepsilon\right). \tag{5a}$$

When $\varepsilon = 0$, we take

(5b) $$dx/d\tau := Ax + g_0(x) = \overline{g}(x) = \overline{f}(x).$$

In the remainder of the proof, we shall assume for simplicity that the g_i are defined and C^{r+2} on all of $\mathbf{R}^n$, that h is defined and C^{r+2} on all of $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$, and that h is T-periodic in t. Also, the term $\varepsilon^{2r+1} h(x, \tau/\varepsilon, \varepsilon)$ will always be taken to be 0 for $\varepsilon = 0$.

Recall that E_- and E_+ denote the stable and unstable subspaces of A. Let π_- and π_+ denote the corresponding projections.

Equation (5) has the bounded solution $x \equiv 0$ for each ε. In order to find the stable manifolds of these solutions, we shall look for solutions of (5) that are bounded for $\tau \geq 0$. Such a solution satisfies, for some $x_0 \in E_-$, the equation

(6) $$\begin{aligned} x(\tau) = e^{A\tau} x_0 &+ \int_0^\tau e^{A(\tau-\sigma)} \pi_- \left(\sum_{i=0}^{2r} \varepsilon^i g_i(x(\sigma)) + \varepsilon^{2r+1} h\left(x(\sigma), \frac{\sigma}{\varepsilon}, \varepsilon\right)\right) d\sigma \\ &+ \int_\infty^\tau e^{A(\tau-\sigma)} \pi_+ \left(\sum_{i=0}^{2r} \varepsilon^i g_i(x(\sigma)) + \varepsilon^{2r+1} h\left(x(\sigma), \frac{\sigma}{\varepsilon}, \varepsilon\right)\right) d\sigma. \end{aligned}$$

See Hale [4] and Chow and Hale [1] for this approach to stable manifold theorems.

We now formulate equation (6) in an abstract manner.

Let $C^0(\mathbf{R}_+, \mathbf{R}^n)$ denote the Banach space of bounded continuous functions from $\mathbf{R}_+ := [0, \infty)$ to $\mathbf{R}^n$ with the sup norm. Define the bounded linear operators $L\colon E_- \to C^0(\mathbf{R}_+, \mathbf{R}^n)$ and $K\colon C^0(\mathbf{R}_+, \mathbf{R}^n) \to C^0(\mathbf{R}_+, \mathbf{R}^n)$ by

$$Lx_0(\tau) := e^{A\tau} x_0;$$
$$Kx(\tau) := \int_0^\tau e^{A(\tau-\sigma)} \pi_- x(\sigma)\, d\sigma + \int_\infty^\tau e^{A(\tau-\sigma)} \pi_+ x(\sigma)\, d\sigma.$$

Define the nonlinear operator $N\colon C^0(\mathbf{R}_+, \mathbf{R}^n) \times \mathbf{R} \to C^0(\mathbf{R}_+, \mathbf{R}^n)$ by

$$N(x,\varepsilon)(\tau) := \begin{cases} \sum_{i=0}^{2r} \varepsilon^i g_i(x(\tau)) + \varepsilon^{2r+1} h(x(\tau), \frac{\tau}{\varepsilon}, \varepsilon), & \varepsilon \neq 0, \\ g_0(x(\tau)), & \varepsilon = 0. \end{cases}$$

Define $F\colon C^0(\mathbf{R}_+, \mathbf{R}^n) \times E_- \times \mathbf{R} \to C^0(\mathbf{R}_+, \mathbf{R}^n)$ by

$$F(x, x_0, \varepsilon) := Lx_0 + KN(x, \varepsilon).$$

Then (6) is equivalent to the fixed point problem $x = F(x, x_0, \varepsilon)$.

If F were C^r and a contraction in x for each (x_0, ε) near $(0,0)$, then the fixed point $x(x_0, \varepsilon)$ would be a C^r function of (x_0, ε) near $(0,0)$, say for $(x_0, \varepsilon) \in (E_- \cap B_\delta) \times (-\delta, \delta)$. The proof of Theorem 1.1 would then be completed as follows. By standard arguments it suffices to find a C^r function $\tilde{\psi}\colon (E_- \cap B_\delta) \times [0, \delta) \to E_+$ such that: (1) for $\varepsilon > 0$, the graph of $\tilde{\psi}(\cdot, \varepsilon)$ is the intersection of a local stable manifold of $\beta_\varepsilon(t)$ with the plane $t = 0$; (2) the graph of $\tilde{\psi}(\cdot, 0)$ is a local stable manifold of 0 for $\dot{z} = \overline{f}(z)$. Define $\chi\colon (E_- \cap B_\delta) \times (-\delta, \delta) \to E_+$ by $\chi(x_0, \varepsilon) = \pi_+ x(x_0, \varepsilon)(0)$. If $x(x_0, \varepsilon)$ were C^r, then χ would be C^r. For each ε, the graph of $\chi(\cdot, \varepsilon)$ would be the intersection of a local stable manifold of $x \equiv 0$ for (5) with the plane $t = 0$. The map $\tilde{\psi}$ is then constructed by tracing through the coordinate changes (for a smaller δ). The restriction $\varepsilon \geq 0$ enters because for $\varepsilon < 0$ the stable manifold of $x \equiv 0$ for (5) corresponds to the unstable manifold of $x \equiv 0$ for (4).

Unfortunately, F is not differentiable because N is not differentiable with respect to ε, even when $\varepsilon \neq 0$. The reason is that

$$\frac{\partial}{\partial \varepsilon} h\left(x(\tau), \frac{\tau}{\varepsilon}, \varepsilon\right) = D_2 h\left(x(\tau), \frac{\tau}{\varepsilon}, \varepsilon\right) \cdot -\frac{\tau}{\varepsilon^2} + D_3 h\left(x(\tau), \frac{\tau}{\varepsilon}, \varepsilon\right),$$

which is not bounded in τ for all $x \in C^0(R_+, R^n)$.

To deal with this difficulty, we shall work in a "scale" of Banach spaces. The idea of doing this comes from Vanderbauwhede and van Gils [**8**], who use this idea to give a nice proof of the C^r center manifold theorem. The same idea was used earlier by Diekmann and van Gils [**2**] and by van Gils [**9**]. There is also some resemblance to the Nash-Moser implicit function theorem [**5**]. For $\gamma > 0$ we define

$$X_\gamma := \{x \in C^0(\mathbf{R}_+, \mathbf{R}^n) \colon e^{\gamma\tau}\|x(\tau)\| \text{ is bounded}\}.$$

Note that $\| \quad \|$ denotes a norm in $\mathbf{R}^n$. In X_γ we use the norm

$$\|x\|_\gamma = \sup_{\tau \geq 0} e^{\gamma\tau}\|x(\tau)\|,$$

which makes X_γ a Banach space.

There exist constants $M > 0$ and $\alpha > 0$ such that

$$\|\pi_- e^{A\tau} x\| \leq M e^{-\alpha\tau}\|x\| \quad \text{for all } x \in \mathbf{R}^n \text{ and for all } \tau \geq 0;$$
$$\|\pi_+ e^{A\tau} x\| \leq M e^{\alpha\tau}\|x\| \quad \text{for all } x \in \mathbf{R}^n \text{ and for all } \tau \leq 0.$$

PROPOSITION 2.3. *If $\alpha > \gamma \geq \eta > 0$, then F maps $X_\gamma \times E_- \times \mathbf{R}$ into X_η.*

Thus if $\alpha > \gamma \geq \eta > 0$, we may define $F_\gamma^\eta \colon X_\gamma \times E_- \times \mathbf{R} \to X_\eta$ by the same formula used to define F.

PROPOSITION 2.4. *If $\gamma \in (0, \alpha)$ and ε is fixed, then $F_\gamma^\gamma(\cdot, \cdot, \varepsilon)$ is C^{r+2}. If $\alpha > \gamma > \eta > 0$, then F_γ^η is C^r.*

PROPOSITION 2.5. *Let $\alpha_1 \in (0, \alpha)$ and $\kappa \in (0, 1)$. There exist $\delta > 0$ and $\delta_1 > 0$ such that if $0 < \gamma < \alpha_1$, $\|x_0\| < \delta$, $|\varepsilon| < \delta$, and $\|x\|_\gamma \leq \delta_1$, then $\|F(x, x_0, \varepsilon)\|_\gamma \leq \delta_1$ and $\|D_1 F_\gamma^\gamma(x, x_0, \varepsilon)\| \leq \kappa$.*

Thus, by Proposition 2.5, $F_\gamma^\gamma(\cdot, x_0, \varepsilon)$ is a contraction of the ball of radius δ_1 in X_γ for $0 < \gamma < \alpha_1$ and $(x_0, \varepsilon) \in B_\delta \times (-\delta, \delta)$. However, F_γ^γ is still not differentiable with respect to ε: the difficulty is the same as before. On the other hand, by Proposition 2.4, if $\gamma > \eta$ then F_γ^η is C^r; but F_γ^η maps X_γ into the larger space X_η. Nevertheless, with Propositions 2.4 and 2.5 in hand, one may appeal to an abstract theorem, due essentially to Vanderbauwhede and van Gils, to conclude that for each $\gamma \in (0, \alpha_1)$, the fixed point of $F_\gamma^\gamma(\cdot, x_0, \varepsilon)$ is a C^r function of (x_0, ε). One can then define the map x as above and complete the proof of Theorem 1.1.

We remark that the reason we assume f is $C^{3(r+1)}$ is as follows: To prove that F_γ^η, $\eta > \gamma$, is C^r away from $\varepsilon = 0$, we need that h is C^{r+2}; to prove this at $\varepsilon = 0$, we need that the power of ε multiplying h in (6) is at least ε^{2r+1}.

In §5 and §6 we study the differentiability of the nonlinear map N, viewed as a map from X_γ to X_η with $\gamma \geq \eta$. In §7 we use the results of this study and some standard arguments to prove Propositions 2.3, 2.4 and 2.5. The abstract theorem just mentioned is stated and proved in §8.

The remaining sections of the paper can be read independently, except that §5 should be read before §6.

3. Sending periodic solutions to zero. In this section we prove Proposition 2.1. Only the assertion that $\beta(t,\varepsilon)$ is C^s requires proof.

Let $\Phi(z,t,\varepsilon)$ be the solution of (1) with $\Phi(z,0,\varepsilon) = z$. Then Φ is C^s and $D_2\Phi(z,t,\varepsilon) = \varepsilon f(\Phi(z,t,\varepsilon),t,\varepsilon)$. Let $\Psi(z,t,\varepsilon) := (1/\varepsilon)[\Phi(z,t,\varepsilon)-z]$, with $\Psi(z,t,0)$ defined by continuity. Now $\Psi(z,0,\varepsilon) \equiv 0$, and $D_2\Psi(z,t,\varepsilon) = (1/\varepsilon)D_2\Phi(z,t,\varepsilon) = f(\Phi(z,t,\varepsilon),t,\varepsilon)$. Therefore

$$\Psi(z,t,\varepsilon) = \int_0^t f(\Phi(z,u,\varepsilon),u,\varepsilon)\,du. \tag{7}$$

Since f and Φ are C^s, Ψ is C^s.

For $\varepsilon \neq 0$, $\Phi(z,T,\varepsilon) = z$ if and only if $\Psi(z,T,\varepsilon) = 0$. We compute from (7) that $\Psi(0,T,0) = 0$ and $D_1\Psi(0,T,0) = TD\overline{f}(0)$, which is invertible. By the implicit function theorem we can solve the equation $\Psi(z,T,\varepsilon) = 0$ near $(z,\varepsilon) = (0,0)$. The solution $z = \alpha(\varepsilon)$ is a C^s function with $\alpha(0) = 0$. Then we define $\beta(t,\varepsilon) := \Phi(\alpha(\varepsilon),t,\varepsilon)$. □

4. Averaging. Proposition 2.2 follows from the following more general formulation.

THEOREM 4.1. *Let $\dot{y} := \varepsilon g(y,t,\varepsilon)$, where $g\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^s and T-periodic in t. Let $1 \leq k < s$. Then there is a T-periodic change of coordinates*

$$y = x + \sum_{i=1}^{k} \varepsilon^i \varphi_i(x,t), \tag{8}$$

valid on a set of the form $B_\rho \times \mathbf{R} \times (-\rho,\rho)$ in $xt\varepsilon$-space, such that in the new coordinates

$$\dot{x} = \sum_{i=1}^{k} \varepsilon^i h_i(x) + \varepsilon^{k+1} h(x,t,\varepsilon).$$

Here φ_i and h_i are C^{s-i+1}, h is C^{s-k} and T-periodic in t, and $h_1(x) = \overline{g}(x)$. Moreover, if $g(0,t,\varepsilon) \equiv 0$, then $h_i(0) = 0$ and $h(0,t,\varepsilon) \equiv 0$.

Proposition 2.2 follows from Theorem 4.1 by setting $k := 2r+1$, $Ax + g_0(x) := h_1(x)$, $g_i(x) := h_{i+1}(x)$ for $1 \leq i \leq 2r$.

PROOF OF THEOREM 4.1. The proof is based on Perko's analysis of high order averaging [**7**].

We begin by computing with equation (8), in which the φ_i are to be chosen later. Differentiation with respect to t gives

$$\dot{y} = \dot{x} + \sum_{i=1}^{k} \varepsilon^i (D_1\varphi_i(x,t)\dot{x} + D_2\varphi_i(x,t)).$$

We set $\dot{y} = \varepsilon g(y,t,\varepsilon)$ and solve for $\dot{x}$:

$$\dot{x} = \left(I + \sum_{i=1}^{k} \varepsilon^i D_1\varphi_i\right)^{-1} \left(\varepsilon g\left(x + \sum_{i=1}^{k} \varepsilon^i \varphi_i, t, \varepsilon\right) - \sum_{i=1}^{k} \varepsilon^i D_2\varphi_i\right) \tag{9}$$

$$= \left(I + \sum_{i=1}^{k-1} \varepsilon^i A_i(x,t) + \varepsilon^k A(x,t,\varepsilon)\right)$$

$$\left(\sum_{i=1}^{k} \varepsilon^i (b_i(x,t) - D_2\varphi_i) + \varepsilon^{k+1} b(x,t,\varepsilon)\right). \tag{10}$$

For each $i = 1, \dots, k-1$,

$$A_i(x,t) = \sum_{i_1+\cdots+i_j=i} a_{i_1\dots i_j}(D_1\varphi_{i_1})\cdots(D_1\varphi_{i_j}),$$

where the coefficients $a_{i_1\dots i_j}$ are not important for our purposes. The b_i are determined by expanding $\varepsilon g(x+\sum_{i=1}^{k} \varepsilon^i\varphi_i, t, \varepsilon)$ in powers of ε. Thus $b_1(x,t) = g(x,t,0)$. For $i = 2, \dots, k$,

$$b_i(x,t) = \sum_{j=1}^{i-1}\sum_{m=j}^{i-1} \frac{1}{j!(i-1-m)!} D_1^j D_3^{i-1-m} g(x,t,0) \sum_{i_1+\cdots+i_j=m} \varphi_{i_1}\cdots\varphi_{i_j}$$
$$+ D_3^{i-1} g(x,t,0).$$

Grouping (10) by powers of ε, we obtain

$$\dot{x} = \sum_{i=1}^{k} \varepsilon^i (c_i(x,t) - D_2\varphi_i) + \varepsilon^{k+1} h(x,t,\varepsilon). \tag{11}$$

Here

$$c_1(x,t) = b_1(x,t) = g(x,t,0),$$

and, for $2 \le i \le k$,

$$c_i(x,t) = \sum_{j=1}^{i-1} A_{i-j}(x,t)(b_j(x,t) - D_2\varphi_j) + b_i(x,t).$$

Thus, for $2 \le i \le k$, c_i is defined in terms of $A_1, \dots, A_{i-1}$, $b_1, \dots, b_i$, which are all defined in terms of $\varphi_1, \dots, \varphi_{i-1}$.

We must now choose T-periodic φ_i so that in (11), the coefficient of ε^i for $1 \le i \le k$ is independent of t. For $i = 1, \dots, k$, define inductively

$$h_i(x) := \bar{c}_i(x,t) := \frac{1}{T}\int_0^T c_i(x,t)\,dt;$$
$$D_2\varphi_i(x,t) := c_i(x,t) - h_i(x); \quad \varphi_i(x,0) := 0.$$

Thus each φ_i is T-periodic in t.

Clearly $c_1(x,t) = g(x,t,0)$ is C^s. The following implications are immediate from the definitions:

$$c_i \text{ is } C^{s-i+1} \Rightarrow h_i \text{ is } C^{s-i+1} \Rightarrow D_2\varphi_i \text{ is } C^{s-i+1}$$
$$\Rightarrow \varphi_i \text{ is } C^{s-i+1} \Rightarrow A_i \text{ is } C^{s-i} \text{ and } b_{i+1} \text{ is } C^{s-i} \Rightarrow c_{i+1} \text{ is } C^{s-i}.$$

(The last two implications hold if $1 \le i \le k-1$, the others if $1 \le i \le k$.) We conclude that for $1 \le i \le k$, b_i, c_i and φ_i are C^{s-i+1}; and for $1 \le i \le k-1$, A_i is C^{s-i}.

We must show that h is C^{s-k}. For this it suffices, by (10) and (11), to show that A and b are C^{s-k}.

We shall say that a function $p(x,t,\varepsilon)$ is C^m_q, $0 \le m \le q$, if $D_1^i D_2^j D_3^l\, p(x,t,\varepsilon)$ exists and is continuous whenever $i+j+l \le q$ and $i+j \le m$. Note that if p is C^m_q, then p is C^m. We shall say that p is C^m_∞ if p is C^m_q for every $q \ge m$.

Now

$$\varphi(x,t,\varepsilon) := x + \sum_{i=1}^{k} \varepsilon^i \varphi_i(x,t)$$

is C^{s-k+1}_∞, and g is C^s. Therefore $g(\varphi(x,t,\varepsilon),t,\varepsilon)$ is C^{s-k+1}_s. (To see this, let $\tilde{g}(x,t,\varepsilon) := g(\varphi(x,t,\varepsilon),t,\varepsilon)$. Note that if $0 \le l \le k-1$, $D_3^l \tilde{g}(x,t,\varepsilon)$ is at least C^{s-k+1}.) Thus, since each $b_i(x,t)$ and $D_2\varphi_i(x,t)$ is at least C^{s-k+1},

$$\varepsilon^k b = g - \sum_{i=1}^{k} \varepsilon^{i-1}(b_i - D_2\varphi_i)$$

is C^{s-k+1}_s. Therefore b is C^{s-k}. (To see this, use the following fact and induction: If $\varepsilon^j c(x,t,\varepsilon)$ is C^m_q, then $\varepsilon^{j-1}c(x,t,\varepsilon)$ is C^m_{q-1} if $q > m$, and is C^{m-1} if $q = m$. To prove this, let $\tilde{c}(x,t,\varepsilon) := \varepsilon^j c(x,t,\varepsilon)$. Then

$$\varepsilon^j c(x,t,\varepsilon) = \tilde{c}(x,t,\varepsilon) = \varepsilon \int_0^1 D_3\tilde{c}(x,t,u\varepsilon)\, du.$$

Therefore

$$\varepsilon^{j-1} c(x,t,\varepsilon) = \int_0^1 D_3\tilde{c}(x,t,u\varepsilon)\, du,$$

which is clearly C^m_{q-1} if $q > m$ and C^{m-1} if $q = m$.)

Moreover, $(I + \sum_{i=1}^k \varepsilon^i D_1\varphi_i(x,t))^{-1}$ is C^{s-k}_∞, and $I + \sum_{i=1}^{k-1} \varepsilon^i A_i(x,t)$ is C^{s-k+1}_∞. Therefore $\varepsilon^k A$ is C^{s-k}_∞, so A is C^{s-k}_∞. It follows that h is C^{s-k}.

Finally we assume that $g(0,t,\varepsilon) \equiv 0$. Clearly $c_1(0,t) \equiv 0$. We have the following implications:

$$c_i(0,t) \equiv 0 \Rightarrow h_i(0) = 0 \Rightarrow D_2\varphi_i(0,t) \equiv 0 \Rightarrow b_{i+1}(0,t) \equiv 0 \Rightarrow c_{i+1}(0,t) \equiv 0.$$

The last two implications hold for $1 \le i \le k-1$, the others for $1 \le k$. Setting $x = 0$ in (9) shows that $\dot{x} = 0$ if $x = 0$. Then (11) implies that $h(0,t,\varepsilon) \equiv 0$. □

5. Differentiability of N for $\varepsilon \neq 0$. Our goal in this section is the following result. Let the spaces X_γ be as in §2.

THEOREM 5.1. *Let $r \ge 1$.*

(1) *Suppose $g\colon \mathbf{R}^n \to \mathbf{R}^n$ is C^r and $g(0) = 0$. Let $\gamma \ge \eta > 0$. Then the mapping $x(\tau) \to g(x(\tau))$ is C^r from X_γ to X_η. Its ith derivative, $i \le i \le r$, at $x \in X_\gamma$ is the mapping $G_\gamma^{\eta(i)}(x)$ defined below.*

(2) *Suppose $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^r, T-periodic in t, and $h(0,t,\varepsilon) \equiv 0$. Let $\gamma \ge \eta > 0$. Then the mapping $x(\tau) \to h(x(\tau), \tau/\varepsilon, \varepsilon)$ is C^r from X_γ to X_η for each*

fixed $\varepsilon \neq 0$. *Its ith derivative,* $1 \leq i \leq r$, *at* $x \in X_\gamma$, *is the mapping* $H_\gamma^{\eta(i,0)}(x,\varepsilon)$ *defined below.*

(3) *Suppose* $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ *is* C^{r+2}, T*-periodic in* t, *and* $h(0,t,\varepsilon) \equiv 0$. *Let* $\gamma > \eta > 0$. *Then the mapping*

$$(x(\tau),\varepsilon) \to h(x(\tau),\tau/\varepsilon,\varepsilon) \tag{12}$$

is C^r *from* $X_\gamma \times (\mathbf{R} \setminus \{0\})$ *to* X_η. *Its* (i,j)*th partial derivative,* $1 \leq i+j \leq r$, *at* (x,ε), *is the mapping* $H_\eta^{\gamma(i,j)}(x,\varepsilon)$ *defined below.*

We shall first give the proof of (3). Then we shall discuss the proofs of (1) and (2) more briefly.

To prove (3) we shall successively prove the following. Let $\gamma > \eta > 0$.

1. The mapping (12) takes $X_\gamma \times (\mathbf{R} \setminus \{0\})$ to X_η, and the mappings $H_\gamma^{\eta(i,j)}(x,\varepsilon)$ are bounded $(i+j)$-multilinear mappings from $X_\gamma^i \times \mathbf{R}^j$ to X_η.
2. Each $H_\gamma^{\eta(i,j)}(x,\varepsilon)$ depends continuously on (x,ε).
3. $H_\gamma^{\eta(i,j)}(x,\varepsilon)$ is the (i,j)th partial derivative of the mapping (12).

Throughout this section and the next two, we use the symbol $\| \ \|$ to denote a norm in $\mathbf{R}^n$ or the operator norm of a linear or multilinear operator between specific Banach spaces. We use $\| \ \|_\gamma$ to denote the norm in X_γ.

LEMMA 5.2. *Suppose* $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ *is* C^1, T*-periodic in* t, *and* $h(0,t,\varepsilon) \equiv 0$. *Let* $x \in X_\gamma$ *and* $\varepsilon \neq 0$. *If* $\gamma \geq \eta > 0$, *then* $h(x(\tau),\tau/\varepsilon,\varepsilon) \in X_\eta$.

PROOF. Since $\gamma \geq 0$ and $x \in X_\gamma$, $\{x(\tau)\colon \tau \geq 0\}$ lies in B_ρ for some $\rho > 0$. Since h is T-periodic in t, $\|D_1 h(x,t,\varepsilon)\|$ is bounded on $B_\rho \times \mathbf{R} \times \{\varepsilon\}$. Let b be a bound. Then for $\tau \geq 0$,

$$e^{\eta\tau}\left\|h\left(x(\tau),\frac{\tau}{\varepsilon},\varepsilon\right)\right\| = e^{\eta\tau}\left\|h\left(x(\tau),\frac{\tau}{\varepsilon},\varepsilon\right) - h\left(0,\frac{\tau}{\varepsilon},\varepsilon\right)\right\| \leq e^{\eta\tau} b\|x(\tau)\| \leq b\|x\|_\gamma. \quad \square$$

Define $\tilde{h}\colon \mathbf{R}^n \times \mathbf{R} \times (\mathbf{R} \setminus \{0\}) \to \mathbf{R}^n$ by

$$\tilde{h}(x,\tau,\varepsilon) := h(x(\tau),\tau/\varepsilon,\varepsilon)$$

LEMMA 5.3. *Suppose* $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ *is* C^{r+2} *and* T*-periodic in* t. *Let* $1 \leq i+j \leq r+2$. *Then for* $\tau \geq 0$,

$$\|D_1^i D_3^j \tilde{h}(x,\tau,\varepsilon)\| \leq b_{ij}(x,\tau,\varepsilon),$$

where $b_{ij}\colon \mathbf{R}^n \times \mathbf{R} \times (\mathbf{R} \setminus \{0\}) \to \mathbf{R}$ *is a polynomial in* τ *of degree* j *whose coefficients are continuous functions of* (x,ε).

PROOF. $D_1^i D_3^j \tilde{h}(x,\tau,\varepsilon)$ is a linear combination of terms of the form

$$D_1^i D_2^k D_3^l h\left(x,\frac{\tau}{\varepsilon},\varepsilon\right)\frac{\partial^{m_1}}{\partial\varepsilon^{m_1}}\left(\frac{\tau}{\varepsilon}\right)\cdots\frac{\partial^{m_k}}{\partial\varepsilon^{m_k}}\left(\frac{\tau}{\varepsilon}\right),$$

where $k + l \leq j$ and $m_1 + \cdots + m_k + l = j$. Note that $\|D_1^i D_2^k D_3^l h(x,t,\varepsilon)\|$ is bounded by a continuous function of (x,ε), and

$$\left|\frac{\partial^m}{\partial\varepsilon^m}\left(\frac{\tau}{\varepsilon}\right)\right| \leq m!\tau\varepsilon^{-(m+1)}. \quad \square$$

PROPOSITION 5.4. *Suppose* $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ *is* C^{r+2} *and* T*-periodic in* t, *and* $\tilde{h}$ *is defined as above. Let* $x \in X_\gamma$ *and* $\varepsilon \neq 0$.

(1) *If* $\gamma \geq \eta > 0$ *and* $1 \leq i \leq r+2$, *there is a bounded i-multilinear map from* X_γ^i *to* X_η *defined by*

$$(x_1, \ldots, x_i) \to D_1^i \tilde{h}(x(\tau), \tau, \varepsilon) x_1(\tau) \cdots x_i(\tau). \tag{13}$$

(2) *If* $\gamma > \eta > 0$, $1 \leq i+j \leq r+2$, *and* $i \geq 1$, *there is a bounded* $(i+j)$*-multilinear map from* $X_\gamma^i \times \mathbf{R}^j$ *to* X_η *defined by*

$$(x_1, \ldots, x_i, \varepsilon_1, \ldots, \varepsilon_j) \to D_1^i D_3^j \tilde{h}(x(\tau), \tau, \varepsilon) x_1(\tau) \cdots x_i(\tau) \varepsilon_1 \cdots \varepsilon_j. \tag{14}$$

(3) *Assume in addition that* $h(0, t, \varepsilon) \equiv 0$. *If* $\gamma > \eta > 0$ *and* $1 \leq j \leq r+1$, *there is a bounded j-multilinear map from* $\mathbf{R}^j$ *to* X_η *defined by*

$$(\varepsilon_1, \ldots, \varepsilon_j) \to D_3^j \tilde{h}(x(\tau), \tau, \varepsilon) \varepsilon_1 \cdots \varepsilon_j. \tag{15}$$

PROOF. Let $\{x(\tau) : \tau \geq 0\} \subset B_\rho$.

(1) Let b be an upper bound for $b_{i0}(x, \tau, \varepsilon)$ on $B_\rho \times \mathbf{R} \times \{\varepsilon\}$. (Recall that $b_{i0}(x, \tau, \varepsilon)$ is constant for fixed (x, ε).) Then for $\tau \geq 0$,

$$\begin{aligned} e^{\eta\tau} \|D_1^i \tilde{h}(x(\tau), \tau, \varepsilon) x_1(\tau) \cdots x_i(\tau)\| &\leq e^{\eta\tau} b \|x_1(\tau)\| \cdots \|x_i(\tau)\| \\ &\leq e^{(\eta - i\gamma)\tau} b \|x_1\|_\gamma \cdots \|x_i\|_\gamma \leq b \|x_1\|_\gamma \cdots \|x_i\|_\gamma. \end{aligned}$$

(2) Let $b(\tau)$ be a polynomial that bounds $b_{ij}(x, \tau, \varepsilon)$ on $B_\rho \times \mathbf{R}_+ \times \{\varepsilon\}$. Then for $\tau \geq 0$,

$$\begin{aligned} &e^{\eta\tau} \|D_1^i D_3^j \tilde{h}(x(\tau), \tau, \varepsilon) x_1(\tau) \cdots x_i(\tau) \varepsilon_1 \ldots \varepsilon_j\| \\ &\quad \leq e^{(\eta - i\gamma)\tau} b(\tau) \|x_1\|_\gamma \cdots \|x_i\|_\gamma |\varepsilon_1| \cdots |\varepsilon_j| \\ &\quad \leq b \|x_1\|_\gamma \cdots \|x_i\|_\gamma |\varepsilon_1| \cdots |\varepsilon_\gamma| \end{aligned}$$

for some constant b, because $i \geq 1$ and $\gamma > \eta$.

(3) Notice that $\tilde{h}(0, \tau, \varepsilon) \equiv 0$, so $D_3^j \tilde{h}(0, \tau, \varepsilon) \equiv 0$ for any j. Let $b(\tau)$ be a polynomial that bounds $b_{ij}(x, \tau, \varepsilon)$ on $B_\rho \times \mathbf{R}_+ \times \{\varepsilon\}$. Then for $\tau \geq 0$,

$$\begin{aligned} &e^{\eta\tau} \|D_3^j \tilde{h}(x(\tau), \tau, \varepsilon)\| \leq e^{\eta\tau} \|D_3^j \tilde{h}(x(\tau), \tau, \varepsilon) - D_3^j \tilde{h}(0, \tau, \varepsilon)\| \\ &\quad \leq e^{\eta\tau} \int_0^1 \|D_1 D_3^j \tilde{h}(ux(\tau), \tau, \varepsilon)\| du \|x(\tau)\| \leq e^{(\eta - \gamma)\tau} b_{ij}(\tau) \|x\|_\gamma \leq b \|x\|_\gamma \end{aligned}$$

for some constant b because $\gamma > \eta$. □

PROPOSITION 5.5. *Assume the hypotheses of Proposition* 5.4.

(1) *If* $\gamma > \eta > 0$ *and* $1 \leq i \leq r+1$, *the i-multilinear map from* X_γ^i *to* X_η *defined by* (13) *depends continuously on* $(x, \varepsilon) \in X_\gamma \times (\mathbf{R} \setminus \{0\})$.

(2) *If* $\gamma > \eta > 0$, $1 \leq i+j \leq r+1$, *and* $i \geq 1$, *the* $(i+j)$*-multilinear map from* $X_\gamma^i \times \mathbf{R}^j$ *to* X_η *defined by* (14) *depends continuously on* $(x, \varepsilon) \in X_\gamma \times (\mathbf{R} \setminus \{0\})$.

(3) *Assume in addition that* $h(0, t, \varepsilon) \equiv 0$. *If* $\gamma > \eta > 0$ *and* $1 \leq j \leq r$, *the j-multilinear map from* $\mathbf{R}^j$ *to* X_η *defined by* (15) *depends continuously on* $(x, \varepsilon) \in X_\gamma \times (\mathbf{R} \setminus \{0\})$.

PROOF. Let $(x, \varepsilon) \in X_\gamma \times (\mathbf{R} \setminus \{0\})$. Choose $\rho > 0$ so that if $0 \leq \tau < \infty$ then $x(\tau) + B_{\varepsilon/2} \subset B_\rho$. Let b be an upper bound for $b_{i+1,0}(\tilde{x}, \tau, \tilde{\varepsilon})$ on $B_\rho \times \mathbf{R} \times [\varepsilon/2, 3\varepsilon/2]$.

Choose a polynomial $c(\tau)$ so that $b_{i1}(\tilde{x},\tau,\tilde{\varepsilon}) \le c(\tau)$ on $B_\rho \times \mathbf{R}_+ \times [\varepsilon/2, 3\varepsilon/2]$. Let $\lambda \in (0,\varepsilon/2)$. If $\|y-x\|_\gamma < \lambda$ and $|\delta-\varepsilon| < \lambda$, then for $\tau \ge 0$:

$$\begin{aligned}
&e^{\eta\tau}\|[D_1^i\tilde{h}(y(\tau),\tau,\delta) - D_1^i\tilde{h}(x(\tau),\tau,\varepsilon)]x_1(\tau)\cdots x_i(\tau)\| \\
&\quad \le e^{\eta\tau}\left\{\int_0^1 \|D_1^{i+1}\tilde{h}(uy(\tau)+(1-u)x(\tau),\tau,u\delta+(1-u)\varepsilon)\|\,du\|y(\tau)-x(\tau)\|\right. \\
&\qquad\qquad \left. + \int_0^1 \|D_1^i D_3\tilde{h}(uy(\tau)+(1-u)x(\tau),\tau,u\delta+(1-u)\varepsilon)\|\,du|\delta-\varepsilon|\right\} \\
&\qquad \times e^{-i\gamma\tau}\|x_1\|_\gamma\cdots\|x_i\|_\gamma \\
&\quad \le e^{\eta\tau}\{be^{-\gamma\tau}\|y-x\|_\gamma + c(\tau)|\delta-\varepsilon|\}e^{-i\gamma\tau}\|x_1\|_\gamma\cdots\|x_i\|_\gamma \\
&\quad \le C\lambda\|x_1\|_\gamma\cdots\|x_i\|_\gamma
\end{aligned}$$

for some constant C because $\gamma > \eta$.

(2) The proof is similar to (1).

(3) Let $b(\tau)$ be a polynomial upper bound for $b_{ij}(\tilde{x},\tau,\tilde{\varepsilon})$ on $B_\rho \times \mathbf{R}_+ \times [\varepsilon/2, 3\varepsilon/2]$. Let $c(\tau)$ be a polynomial upper bound for $b_{1,j+1}(\tilde{x},\tau,\tilde{\varepsilon})$ on $B_\rho \times \mathbf{R}_+ \times [\varepsilon/2, 3\varepsilon/2]$. Let $\lambda \in (0,\varepsilon/2)$. If $\|y-x\|_\gamma < \lambda$ and $|\delta-\varepsilon| < \lambda$, then for $\tau \ge 0$:

$$\begin{aligned}
&e^{\eta\tau}\|D_3^j\tilde{h}(y(\tau),\tau,\delta) - D_3^j\tilde{h}(x(\tau),\tau,\varepsilon)\| \\
&\quad \le e^{\eta\tau}\left\{\int_0^1 \|D_1D_3^j\tilde{h}(uy(\tau)+(1-u)x(\tau),\tau,u\delta+(1-u)\varepsilon)\|du\|y(\tau)-x(\tau)\|\right. \\
&\qquad\qquad \left. + \int_0^1 \|D_3^{j+1}\tilde{h}(uy(\tau)+(1-u)x(\tau),\tau,u\delta+(1-u)\varepsilon)\|du|\delta-\varepsilon|\right\} \\
&\quad \le e^{\eta\tau}\left\{b(\tau)e^{-\gamma\tau}\|y-x\|_\gamma\right. \\
&\qquad\qquad + \int_0^1\int_0^1 \|D_1D_3^{j+1}\tilde{h}(vuy(\tau)+v(1-u)x(\tau),\tau,u\delta+(1-u)\varepsilon)\| \\
&\qquad\qquad\qquad \left. \cdot\|uy(\tau)+(1-u)x(\tau)\|\,dv\,du|\delta-\varepsilon|\right\} \\
&\quad \le e^{\eta\tau}\{b(\tau)e^{-\gamma\tau} + c(\tau)e^{-\gamma\tau}\rho\}\lambda \le C\lambda
\end{aligned}$$

for some constant C. □

Let $\gamma > \eta > 0$. Suppose $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^{r+2}, T-periodic in t, and $h(0,t,\varepsilon) \equiv 0$. Then by Lemma 5.2 we may define $H_\gamma^\eta\colon X_\gamma \times (\mathbf{R}\setminus\{0\}) \to X_\eta$ by

$$H_\gamma^\eta(x,\varepsilon)(\tau) := h(x(\tau),\tau/\varepsilon,\varepsilon) = \tilde{h}(x(\tau),\tau,\varepsilon).$$

If $1 \le i+j \le r$, $x \in X_\gamma$, and $\varepsilon \ne 0$, we define $H_\gamma^{\eta(i,j)}(x,\varepsilon)$ to be the bounded $(i+j)$-multilinear map from $X_\gamma^i \times \mathbf{R}^j$ to X_η given by

$$\begin{aligned}
&H_\gamma^{\eta(i,j)}(x,\varepsilon)(x_1,\dots,x_i,\varepsilon_1,\dots,\varepsilon_j)(\tau) \\
&\qquad := D_1^iD_3^j\tilde{h}(x(\tau),\tau,\varepsilon)(x_1(\tau),\dots,x_i(\tau),\varepsilon_1,\dots,\varepsilon_\gamma).
\end{aligned}$$

This makes sense by Proposition 5.4.

We shall now complete the proof of Theorem 5.1(3). For uniformity of exposition we define $H_\gamma^{\eta(0,0)} := H$.

Consider a pair (i,j) with $0 \le i+j \le r-1$. Fix $x \in X_\gamma$ and let $y \in X_\gamma$ be close to x. We must show

$$H_\gamma^{\eta(i,j)}(y,\varepsilon) - H_\gamma^{\eta(i,j)}(x,\varepsilon) - H_\gamma^{\eta(i+1,j)}(x,\varepsilon)(y-x) = o(\|y-x\|_\gamma); \tag{16}$$

$$H_\gamma^{\eta(i,j)}(x,\delta) - H_\gamma^{\eta(i,j)}(x,\varepsilon) - H_\gamma^{\eta(i,j+1)}(x,\varepsilon)(\delta-\varepsilon) = o(|\delta-\varepsilon|). \tag{17}$$

For then $D_1 H_\gamma^{\eta(i,j)}(x,\varepsilon) = H_\gamma^{\eta(i+1,j)}(x,\varepsilon)$, and $D_2 H_\gamma^{\eta(i,j)}(x,\varepsilon) = H_\gamma^{\eta(i,j+1)}(x,\varepsilon)$. Since $H_\gamma^{\eta(i+1,j)}$ and $H_\gamma^{\eta(i,j+1)}$ are continuous by Proposition 5.5, $H_\gamma^{\eta(i,j)}$ is continuously differentiable for $0 \le i+j \le r-1$, and its partial derivatives are as stated. Thus Theorem 5.1(3) is proved.

Let $S = \{x_1,\dots,x_i,\varepsilon_1,\dots,\varepsilon_j\} \in X_\gamma^i \times \mathbf{R}^j \colon \|x_1\|_\gamma = \cdots = \|x_i\|_\gamma = |\varepsilon_1| = \cdots |\varepsilon_j| = 1\}$. To show (16), we compute

$$\begin{aligned}
&\|H_\gamma^{\eta(i,j)}(y,\varepsilon) - H_\gamma^{\eta(i,j)}(x,\varepsilon) - H_\gamma^{\eta(i+1,j)}(x,\varepsilon)(y-x)\| \\
&\quad = \sup_S \|[H_\gamma^{\eta(i,j)}(y,\varepsilon) - H_\gamma^{\eta(i,j)}(x,\varepsilon) - H_\gamma^{\eta(i+1,j)}(x,\varepsilon)(y-x)] \\
&\qquad\qquad \cdot (x_1,\dots,x_i,\varepsilon_1,\dots,\varepsilon_\gamma)\|_\eta \\
&\quad \le \sup_S \sup_{\tau \ge 0} e^{\eta\tau} \left\| \int_0^1 [D_1^{i+1} D_3^j \tilde{h}(uy(\tau) + (1-u)x(\tau),\tau,\varepsilon) - D_1^{i+1} D_3^j \tilde{h}(x(\tau),\tau,\varepsilon)] \right. \\
&\qquad\qquad \left. \cdot (y(\tau)-x(\tau), x_1(\tau),\dots,x_i(\tau),\varepsilon_1,\dots,\varepsilon_j)\,du \right\| \\
&\quad \le \sup_{0 \le u \le 1} \|H_\gamma^{\eta(i+1,j)}(uy + (1-u)x,\varepsilon) - H_\gamma^{\eta(i+1,j)}(x,\varepsilon)\| \, \|y-x\|_\gamma.
\end{aligned}$$

Since $H_\gamma^{\eta(i+1,j)}$ depends continuously on (x,ε), this expression can be made an arbitrarily small multiple of $\|y-x\|_\gamma$ by taking $\|y-x\|_\gamma$ sufficiently small.

The proof of (17) is similar.

To prove Theorem 5.1(2), assume the hypotheses and let $\gamma \ge \eta > 0$. We define H_γ^η by the same formula used previously; we may do this for $\gamma \ge \eta$ by Lemma 5.2. We also define $H_\gamma^{\eta(i,0)}(x,\varepsilon)$, $1 \le i \le r$, by the formula used previously; Proposition 5.4(1) shows we may do this for $\gamma \ge \eta$ and $1 \le i \le r$. To see that $H_\gamma^{\eta(i,0)}(x,\varepsilon)$ depends continuously on x for fixed ε, we appeal to:

PROPOSITION 5.6. *Suppose $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^r and T-periodic in t. If $\gamma \ge \eta > 0$ and $1 \le i \le r$, then $H_\gamma^{\eta(i,0)}(x,\varepsilon)$ depends continuously on $x \in X_\gamma$ for fixed ε.*

PROOF. Fix $x \in X_\gamma$, $\varepsilon \ne 0$, and i with $1 \le i \le r$. Choose $\rho > 0$ so that B_ρ contains $\{x(\tau)\colon \tau \ge 0\}$ in its interior. Since h is T-periodic in t, $D_1^i h$ is uniformly continuous on $B_\rho \times \mathbf{R} \times \{\varepsilon\}$. Choose $\lambda > 0$. Then there exists $\delta > 0$ such that if $\tau \ge 0$ and $\|z - x(\tau)\| < \delta$ then $z \in B_\rho$ and $\|D_1^i h(z,t,\varepsilon) - D_1^i h(x(\tau),t,\varepsilon)\| < \lambda$. Now let $y \in X_\gamma$ with $\|y - x\| < \delta$. Then $\|y(\tau) - x(\tau)\| < \delta$ for all $\tau \ge 0$. If $\gamma \ge \eta > 0$, we have

$$\begin{aligned}
&e^{\eta\tau}\|[D_1^i \tilde{h}(y(\tau),\tau,\varepsilon) - D_1^i \tilde{h}(x(\tau),\tau,\varepsilon)](x_1(\tau),\dots,x_i(\tau))\| \\
&\qquad < e^{\eta\tau}\lambda e^{-i\gamma\tau}\|x_1\|_\gamma \cdots \|x_i\|_\gamma \le \lambda \|x_1\|_\gamma \cdots \|x_i\|_\gamma. \quad \square
\end{aligned}$$

The proof of Theorem 5.1(2) is now similar to the proof of Theorem 5.1(3) above.

To prove Theorem 5.1(1), we define $G^{\eta}_{\gamma}\colon X_{\gamma} \to X_{\eta}$, $\gamma \geq \eta$, by $G^{\eta}_{\gamma}(x)(\tau) := g(x(\tau))$, and we define $G^{\eta(i)}_{\gamma}(x)\colon X^i_{\gamma} \to X_{\eta}$ by

$$G^{\eta(i)}_{\gamma}(x)(x_1,\dots,x_i)(\tau) = D^i g(x(\tau))(x_1(\tau),\dots,x_i(\tau)).$$

The proof is similar to that of Theorem 5.1(2).

6. Differentiability of N for $\varepsilon = 0$. Our goal in this section is the following result. Assume $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^{r+2}, T-periodic in t, and $h(0,t,\varepsilon) \equiv 0$. For $\gamma > \eta > 0$ define $\tilde{H}^{\eta}_{\gamma}\colon X_{\gamma} \times \mathbf{R} \to X_{\eta}$ by

$$\tilde{H}^{\eta}_{\gamma}(x,\tau) := \begin{cases} \varepsilon^{2r+1} h(x(\tau),\tau/\varepsilon,\varepsilon), & \varepsilon \not\equiv 0; \\ 0, & \varepsilon = 0. \end{cases}$$

THEOREM 6.1. *Under the above assumptions, $\tilde{H}^{\eta}_{\gamma}$ is C^r.*

PROOF. Theorem 5.1(3) implies that $\tilde{H}^{\eta}_{\gamma}$ is C^r on $X_{\gamma} \times (\mathbf{R} \setminus \{0\})$. Thus we must study $\tilde{H}^{\eta}_{\gamma}$ at points $(x,0)$.

Recall that $\tilde{h}(x,\tau,\varepsilon) := h(x,\tau/\varepsilon,\varepsilon)$. For $1 \leq i+j \leq r$ and $(x,\varepsilon) \in X_{\gamma} \times (\mathbf{R}\setminus\{0\})$, define $\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon)$ to be the bounded $(i+j)$-multilinear map from $X^i_{\gamma} \times \mathbf{R}^j$ to X_{η} given by

$$(x_1(\tau),\dots,x_i(\tau),\varepsilon_1,\dots,\varepsilon_j) \to D^i_1 D^j_3[\varepsilon^{2r+1}\tilde{h}(x(\tau),\tau,\varepsilon)](x_1(\tau),\dots,x_i(\tau),\varepsilon_1,\dots,\varepsilon_j).$$

Define $\tilde{H}^{\eta(i,j)}_{\gamma}(x,0)$ to be the zero map. For uniformity of exposition we define $\tilde{H}^{\eta(0,0)}_{\gamma} := \tilde{H}^{\eta}_{\gamma}$.

To prove Theorem 6.2, we shall show:

1. $\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon)$ depends continuously on (x,ε) for $1 \leq i+j \leq r$.
2. $D_1\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon) = \tilde{H}^{\eta(i+1,j)}_{\gamma}(x,\varepsilon)$ and $D_2\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon) = \tilde{H}^{\eta(i,j+1)}_{\gamma}(x,\varepsilon)$ for $0 \leq i+j \leq r-1$.

These assertions need only be proved at points $(x,0)$.

LEMMA 6.2. *Suppose $h\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^n$ is C^{r+2} and T-periodic in t. Let $1 \leq i+j \leq r+2$. Let $\rho > 0$. Then there is a polynomial $b_{ij}(\tau)$ of degree j such that for all $x \in B_{\rho}$ and for all $\varepsilon \neq 0$ sufficiently small, $\|D^i_1 D^j_3[\varepsilon^{2r+1}\tilde{h}(x,\tau,\varepsilon)]\|$ is bounded by $\varepsilon^{2r+1-2j} b_{ij}(\tau)$ for $\tau \geq 0$.*

The proof is an exercise in differentiation; see the proof of Lemma 5.2. Notice that if $j \leq r$, then $2r+1-2j \geq 1$.

We shall show that $\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon)$ depends continuously on (x,ε) at $\varepsilon = 0$ by showing that $\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly on bounded subsets of X_{γ}. Let $x \in B_{\rho}$ and let $\varepsilon \neq 0$ be small. If $i \geq 1$, then

$$\begin{aligned} (18)\quad & \|\tilde{H}^{\eta(i,j)}_{\gamma}(x,\varepsilon)(x_1,\dots,x_i,\varepsilon_1,\dots,\varepsilon_j)\|_{\eta} \\ &= \sup_{\tau\geq 0} e^{\eta\tau}\|D^i_1 D^j_3[\varepsilon^{2r+1}\tilde{h}(x(\tau),\tau,\varepsilon)](x_1(\tau),\dots,x_i(\tau),\varepsilon_1,\dots,\varepsilon_j)\| \\ &\leq \sup_{\tau\geq 0} e^{\eta\tau}\varepsilon^{2r+1-2j} b_{ij}(\tau) e^{-i\gamma\tau}\|x_1\|_{\gamma}\cdots\|x_i\|_{\gamma}|\varepsilon_1|\cdots|\varepsilon_{\gamma}|, \end{aligned}$$

and $\sup_{\tau\geq 0} e^{\eta\tau}\varepsilon^{2r+1-2j}b_{ij}(\tau)e^{-i\gamma\tau} \to 0$ as $\varepsilon \to 0$. If $i = 0$, we use the fact that $\tilde{h}(0,\tau,\varepsilon) \equiv 0$:

$$(19)\qquad \begin{aligned} \|\tilde{H}_\gamma^{\eta(0,j)}(x,\varepsilon)\|_\eta &= \sup_{\tau\geq 0} e^{\eta\tau}\|D_3^j[\varepsilon^{2r+1}\tilde{h}(x(\tau),\tau,\varepsilon)]\| \\ &= \sup_{\tau\geq 0} e^{\eta\tau}\left\| D_3^j\left[\varepsilon^{2r+1}\int_0^1 D_1\tilde{h}(ux(\tau),\tau,\varepsilon)\,du\right]x(\tau)\right\| \\ &\leq \sup_{\tau\geq 0}\sup_{0\leq u\leq 1} e^{\eta\tau}\|D_1D_3^j[\varepsilon^{2r+1}\tilde{h}(ux(\tau),\tau,\varepsilon)]\|\,\|x(\tau)\| \\ &\leq \sup_{\tau\geq 0} e^{\eta\tau}\varepsilon^{2r+1-2j}b_{1j}(\tau)e^{-\gamma\tau}\|x\|_\gamma, \end{aligned}$$

which approaches 0 as ε approaches 0 uniformly on bounded subsets of X_γ.

Finally we must compute the partial derivatives of $\tilde{H}_\gamma^{\eta(i,j)}$ at $(x,0)$, $0 \leq i+j \leq r-1$. It is clear that $D_1\tilde{H}_\gamma^{\eta(i,j)}(x,0) = \tilde{H}_\gamma^{\eta(i+1,j)}(x,0)$ since $\tilde{H}_\gamma^{\eta(i,j)}(x,0) = 0$ for all x and $\tilde{H}_\gamma^{\eta(i+1,j)}(x,0) = 0$. To show that $D_2\tilde{H}_\gamma^{\eta(i,j)}(x,0) = \tilde{H}_\gamma^{\eta(i,j+1)}(x,0) = 0$, we must show that for $0 \leq i+j \leq r-1$, $\tilde{H}_\gamma^{\eta(i,j)}(x,\varepsilon) = o(|\varepsilon|)$ for fixed x. This follows from (18) and (19) by noting that for $j \leq r-1$, $2r+1-2j \geq 3$. □

7. Proofs of Propositions 2.3, 2.4 and 2.5. We consider the mapping $F(x,x_0,\varepsilon) := Lx_0 + KN(x,\varepsilon)$, where L, K, and N are as defined in §2. The numbers M and α and the spaces X_γ are also as defined in §2. We assume for simplicity that the norms of the projections π_+ and π_- are one; of course, this can always be arranged by a linear change of coordinates.

LEMMA 7.1. *If $0 < j \leq \alpha$, then $\|Lx_0\|_\gamma \leq M\|x_0\|$.*

PROOF.

$$\|Lx_0\|_\gamma = \sup_{\tau\geq 0} e^{\gamma\tau}\|e^{A\tau}x_0\| \leq \sup_{\tau\geq 0} e^{\gamma\tau}Me^{-\alpha\tau}\|x_0\| \leq M\|x_0\|. \quad \square$$

LEMMA 7.2. *If $\alpha > \gamma \geq \eta > 0$, then K maps X_γ into X_η. The norm of K as a linear map from X_γ to X_η is at most $M(1/(\alpha-\gamma) + 1/(\alpha+\gamma))$.*

PROOF. $\|Kx\|_\eta = \sup_{\tau\geq 0} e^{\eta\tau}\|Kx(\tau)\|$, and

$$\begin{aligned} e^{\eta\tau}\|Kx(\tau)\| &\leq e^{\eta\tau}\left\{\int_0^\tau Me^{-\alpha(\tau-\sigma)}e^{-\gamma\sigma}\|x\|_\gamma\,d\sigma + \int_\tau^\infty Me^{\alpha(\tau-\sigma)}e^{-\gamma\sigma}\|x\|_\gamma\,d\sigma\right\} \\ &= Me^{\eta\tau}\left\{\frac{e^{-\gamma\tau}-e^{-\alpha\tau}}{\alpha-\gamma} + \frac{e^{-\gamma\tau}}{\alpha+\gamma}\right\}\|x\|_\gamma \leq M\left(\frac{1}{\alpha-\gamma}+\frac{1}{\alpha+\gamma}\right)\|x\|_\gamma \end{aligned}$$

since $\alpha > \gamma \geq \eta$. □

Now Theorem 5.1(1) and (2) imply in particular that N maps X_γ into X_η if $\gamma \geq \eta$. Proposition 2.3 follows from this fact together with Lemmas 7.1 and 7.2.

Proposition 2.4 follows from Lemma 7.2, Proposition 2.2, Theorem 5.1(1) and (2), and Theorem 6.1.

To prove Proposition 2.5, let $\alpha_1 \in (0,\alpha)$ and let $M_1 = M(1/(\alpha-\alpha_1) + 1/\alpha)$. Choose $\delta > 0$ so small that if $\|y\| < \delta$ and $|\varepsilon| < \delta$, then

$$(20)\qquad \sum_{i=0}^{2r}|\varepsilon|^i\|Dg_i(y)\| + |\varepsilon|^{2r+1}\|D_1h(y,t,\varepsilon)\| < \min\left(\frac{1}{2M_1}, \frac{\kappa}{M_1}\right).$$

We can do this because $Dg_0(0) = 0$. If $\|x\|_\gamma < \delta$ and $|\varepsilon| < \delta$, then

$$\begin{aligned}\|N(x,\varepsilon)\|_\gamma &\le \sup_{\tau\ge 0} e^{\gamma\tau}\left(\sum_{i=0}^{2r}|\varepsilon|^i\|g_i(x(\tau))\| + |\varepsilon|^{2r+1}\left\|h\left(x(\tau),\frac{\tau}{\varepsilon},\varepsilon\right)\right\|\right)\\ &\le \sup_{\tau\ge 0} e^{\gamma\tau}\left(\sum_{i=0}^{2r}|\varepsilon|^i\int_0^1\|Dg_i(ux(\tau))\|\,du\right.\\ &\qquad\left. + |\varepsilon|^{2r+1}\int_0^1\left\|D_1h\left(ux(\tau),\frac{\tau}{\varepsilon},\varepsilon\right)\right\|\,du\right)\|x(\tau)\|\\ &\le \frac{\delta}{2M_1}.\end{aligned}$$

If $\gamma \in (0,\alpha_1)$, $\|x_0\| < \delta/2M$, $|\varepsilon| < \delta$, and $\|x\|_\gamma \le \delta$, then this computation and Lemmas 7.1 and 7.2 imply that

$$\|F(x,x_0,\varepsilon)\|_\gamma \le \|Lx_0\| + \|KN(x,\varepsilon)\| \le M\delta/2M + M_1\delta/2M_1 = \delta.$$

Thus we take $\delta_1 := \min(\delta/2M,\delta)$. The estimate on $\|D_1F_\gamma^\gamma(x,x_0,\varepsilon)\|$ uses (20) and the formula

$$[D_1N_\gamma^\gamma(x,x_0,\varepsilon)x_1](\tau) = \left[\sum_{i=0}^{2r}\varepsilon^i Dg_i(x(\tau)) + \varepsilon^{2r+1}D_1h\left(x(\tau),\frac{\tau}{\varepsilon},\varepsilon\right)\right]x_1(\tau),$$

which follows from Theorems 5.1(1) and (2).

8. Contractions on a scale of Banach spaces. A *scale* of Banach spaces is a family of Banach spaces $\{X_\gamma\}$, $\gamma_0 < \gamma < \gamma_1$, and one-to-one bounded linear maps $J_\gamma^\eta\colon X_\gamma \to X_\eta$, $\gamma > \eta$, such that $J_\gamma^\eta \circ J_\xi^\gamma = J_\xi^\eta$ whenever $\xi > \gamma > \eta$. For example, the X_γ defined in §2, $0 < \gamma < \infty$, together with the natural injections J_γ^η of X_γ into X_η for $\gamma > \eta$, form a scale of Banach spaces.

Let Λ be an open subset of some Banach space. A family of mappings $F_\gamma^\eta\colon X_\gamma \times \Lambda \to X_\eta$, $\gamma \ge \eta$, is called *scale invariant* if $J_\gamma^\eta \circ F_\xi^\gamma = F_\xi^\eta$ whenever $\xi \ge \gamma > \eta$, and $F_\gamma^\eta(J_\xi^\gamma x,\lambda) = F_\xi^\eta(x,\lambda)$ whenever $\xi > \gamma \ge \eta$.

THEOREM 8.1. *Let $\{X_\gamma\}$ be a scale of Banach spaces and $\{F_\gamma^\eta\}$ a scale invariant family of mappings. For each γ let Q_γ be a closed convex subset of X_γ such that $J_\gamma^\eta Q_\gamma \subset Q_\eta$ for $\gamma > \eta$, and $F_\gamma^\gamma(Q_\gamma \times \Lambda) \subset Q_\gamma$ for all γ. Assume:*

(1) $D_1F_\gamma^\gamma(x,\lambda)$ exists for all γ and all $(x,\lambda) \in Q_\gamma \times \Lambda$.

(2) There exists a number κ, $0 < \kappa < 1$, such that $\|D_1F_\gamma^\gamma(x,\lambda)\| \le \kappa < 1$ for all γ and all $(x,\lambda) \in Q_\gamma \times \Lambda$.

(3) F_γ^η is C^r, $r \ge 1$, if $\gamma > \eta$.

Let $x_\gamma(\lambda)$ denote the unique fixed point of $F_\gamma^\gamma(\cdot,\lambda)$ in Q_γ. Then $J_\gamma^\eta x_\gamma(\lambda) = x_\eta(\lambda)$ whenever $\gamma > \eta$, and $x_\gamma(\lambda)$ is C^r for each γ. Moreover, $Dx_\gamma(\lambda)$ is the unique fixed point of the equation

$$A = D_1F_\gamma^\gamma(x_\gamma(\lambda),\lambda)A + D_2F_\xi^\gamma(x_\xi(\lambda),\lambda)$$

for any $\xi > \gamma$.

In using this theorem to prove Theorem 1.1, one refers to Proposition 2.4: X_γ and F_γ^η are as in §2, $(\gamma_0,\gamma_1) := (0,\alpha_1)$, Λ is the ball of radius δ_1 in $E_- \times \mathbf{R}$, and Q_γ

is the ball of radius δ in X_γ. Propositions 2.3, 2.4 and 2.5 say that the hypotheses of Theorem 8.1 are satisfied.

In stating Theorem 8.1, we have made no attempt to give the weakest hypotheses that permit the conclusion that some $x_\gamma(\lambda)$ is C^r. In particular, Theorem 8.1 as stated does not apply to the proof of the center manifold theorem [**8**]: there F_γ^η is C^r only for γ sufficiently greater than η. All the ideas necessary to prove Theorem 8.1 are in [**8**]. We include the proof for the reader's convenience.

In the proof, $L^k(X,Y)$ denotes the Banach space of k-multilinear maps between Banach spaces X and Y, with the usual operator norm. The symbol $\| \quad \|$ is used to denote the norm in any Banach space, including the spaces $L^k(X,Y)$. Which norm is meant should always be clear from the expression between the vertical lines.

PROOF. For each $\gamma \in (\gamma_0, \gamma_1)$ and each $\lambda \in \Lambda$, $F_\gamma^\gamma | Q_\gamma \times \{\lambda\}$ is a contraction with contraction constant κ. By the contraction mapping theorem, $F_\gamma^\gamma | Q_\gamma \times \{\lambda\}$ has a unique fixed point $x_\gamma(\lambda)$. The first part of the conclusion follows from

$$F_\eta^\eta(J_\gamma^\eta x_\gamma(\lambda), \lambda) = F_\gamma^\eta(x_\gamma(\lambda), \lambda) = J_\gamma^\eta \circ F_\gamma^\gamma(x_\gamma(\lambda), \lambda) = J_\gamma^\eta x_\gamma(\lambda).$$

We shall now prove the second part of the conclusion in the case $r = 1$ in a sequence of steps.

1. The map $x_\eta(\lambda)$ is Lipschitz continuous for all η. Let $\gamma > \eta$. Then

$$\begin{aligned}\|x_\eta(\mu) - x_\eta(\lambda)\| &= \|F_\eta^\eta(x_\eta(\mu), \mu) - F_\eta^\eta(x_\eta(\lambda), \lambda)\| \\ &\leq \|F_\eta^\eta(x_\eta(\mu), \mu) - F_\eta^\eta(x_\eta(\lambda), \mu)\| \\ &\quad + \|F_\eta^\eta(x_\eta(\lambda), \mu) - F_\eta^\eta(x_\eta(\lambda), \lambda)\| \\ &\leq \kappa\|x_\eta(\mu) - x_\eta(\lambda)\| + \|F_\gamma^\eta(x_\gamma(\lambda), \mu) - F_\gamma^\eta(x_\gamma(\lambda), \lambda)\|.\end{aligned}$$

(To justify the last step, note that $F_\eta^\eta(x_\eta(\lambda), \mu) = F_\eta^\eta(J_\gamma^\eta x_\gamma(\lambda), \mu) = F_\gamma^\eta(x_\gamma(\lambda), \mu)$.)

Therefore

$$\|x_\eta(\mu) - x_\eta(\lambda)\| \leq (1-\kappa)^{-1} \int_0^1 \|D_2 F_\gamma^\eta(x_\gamma(\lambda), u\mu + (1-u)\lambda)\| \, du |\mu - \lambda|.$$

Since F_γ^η is C^1, for fixed λ there is a constant $b(\lambda)$ such that

$$\|D_2 F_\gamma^\eta(x_\gamma(\lambda), u\mu + (1-u)\lambda)\| \leq b(\lambda)$$

for all μ sufficiently near λ and for all $u \in [0,1]$. Therefore $\|x_\eta(\mu) - x_\eta(\lambda)\| \leq b(\lambda)(1-\kappa)^{-1}|\mu - \lambda|$ for all μ sufficiently near λ.

2. For fixed λ and $\xi > \gamma$, consider the equation

$$A = D_1 F_\gamma^\gamma(x_\gamma(\lambda), \lambda)A + D_2 F_\xi^\gamma(x_\xi(\lambda), \lambda), \tag{21}$$

where $A \in L(\Lambda, X_\gamma)$.

We note three facts:

(a) $D_2 F_\xi^\gamma(x_\xi(\lambda), \lambda)$ is independent of ξ.

(b) $J_\gamma^\eta D_1 F_\gamma^\gamma(x_\gamma(\lambda), \lambda) = D_1 F_\eta^\eta(x_\eta(\lambda), \lambda) J_\gamma^\eta$.

(c) $J_\gamma^\eta D_2 F_\xi^\gamma(x_\xi(\lambda), \lambda) = D_2 F_\xi^\eta(x_\xi(\lambda), \lambda)$.

The second follows from the chain rule:

$$J_\gamma^\eta D_1 F_\gamma^\gamma(x, \lambda) = D_1 J_\gamma^\eta F_\gamma^\gamma(x, \lambda) = D_1 F_\gamma^\eta(x, \lambda) = D_1 F_\eta^\eta(J_\gamma^\eta x, \lambda) J_\gamma^\eta.$$

The third fact also follows from the chain rule. The first follows from differentiation with respect to λ of the identity

$$F_\xi^\gamma(x_\xi(\lambda),\lambda) = F_\omega^\gamma(x_\omega(\lambda),\lambda), \qquad \omega > \xi > \gamma.$$

Note that

$$D_1F_\xi^\gamma(x_\xi(\lambda),\lambda)\frac{dx_\xi}{d\lambda} = D_1F_\xi^\gamma(x_\xi(\lambda),\lambda)J_w^\xi\frac{dx_\omega}{d\lambda} = D_1F_w^\gamma(x_w(\lambda),\lambda)\frac{dx_w}{d\lambda}.$$

Now the equation (21), which is linear in A, has the unique fixed point $A_\gamma(\lambda)$. By (a), $A_\gamma(\lambda)$ does not depend on ξ. Moreover, applying J_γ^n to both sides of (21), we find, using (b) and (c):

$$J_\gamma^\eta A_\gamma(\lambda) = D_1F_\eta^\eta(x_\eta(\lambda),\lambda)J_\gamma^\eta A_\gamma(\lambda) + D_2F_\xi^\eta(x_\xi(\lambda),\lambda).$$

Therefore $A_\eta(\lambda) = J_\gamma^\eta A_\gamma(\lambda)$.

3. We shall show that $x_\eta(\lambda)$ is differentiable with derivative $A_\eta(\lambda)$. Let $\xi > \gamma > \eta$. Then

$$\begin{aligned}
&x_\eta(\mu) - x_\eta(\lambda) - A_\eta(\lambda)(\mu-\lambda)\\
&\quad = J_\gamma^\eta F_\gamma^\gamma(x_\gamma(\mu),\mu) - J_\gamma^\eta F_\gamma^\gamma(x_\gamma(\lambda),\lambda) - J_\gamma^\eta A_\gamma(\lambda)(\mu-\lambda)\\
&\quad = \{J_\gamma^\eta F_\gamma^\gamma(x_\gamma(\mu),\mu) - J_\gamma^\eta F_\gamma^\gamma(x_\gamma(\lambda),\mu) - J_\gamma^\eta D_1F_\gamma^\gamma(x_\gamma(\lambda),\lambda)A_\gamma(\lambda)(\mu-\lambda)\}\\
&\qquad + J_\gamma^\eta\{F_\xi^\gamma(x_\xi(\lambda),\mu) - F_\xi^\gamma(x_\xi(\lambda),\lambda) - D_2F_\xi^\gamma(x_\xi(\lambda),\lambda)(\mu-\lambda)\}\\
&\quad = \{F_\gamma^\eta(x_\gamma(\mu),\mu) - F_\gamma^\eta(x_\gamma(\lambda),\mu) - D_1F_\gamma^\eta(x_\gamma(\lambda),\lambda)A_\gamma(\lambda)(\mu-\lambda)\}\\
&\qquad + J_\gamma^\eta\{\cdots\} = D_1F_\gamma^\eta(x_\gamma(\lambda),\lambda)(x_\gamma(\mu) - x_\gamma(\lambda) - A_\gamma(\lambda)(\mu-\lambda)) + R(\lambda,\mu)\\
&\quad = D_1F_\eta^\eta(x_\eta(\lambda),\lambda)(x_\eta(\mu) - x_\eta(\lambda) - A_\eta(\lambda)(\mu-\lambda)) + R(\lambda,\mu).
\end{aligned}$$

Here

$$\begin{aligned}
R(\lambda,\mu) = \int_0^1 &[D_1F_\gamma^\eta(ux_\gamma(\mu) + (1-u)x_\gamma(\lambda),\mu)\\
&\qquad - D_1F_\gamma^\eta(x_\gamma(\lambda),\lambda)](x_\gamma(\mu) - x_\gamma(\lambda))\,du\\
&+ J_\gamma^\eta\int_0^1 [D_2F_\xi^\gamma(x_\xi(\lambda),u\mu + (1-u)\lambda) - D_2F_\xi^\gamma(x_\gamma(\lambda),\lambda)](\mu-\lambda)\,du.
\end{aligned}$$

Since x_γ, x_ξ, $D_1F_\gamma^\eta$, and $D_2F_\xi^\gamma$ are continuous, given $\varepsilon > 0$ and λ there is a $\delta > 0$ such that if $|\mu-\lambda| < \delta$ and $0 \le u \le 1$, then the norms of the expressions in square brackets are less than ε. Moreover, for all sufficiently small δ, $\|x_\gamma(\mu) - x_\gamma(\lambda)\| \le c|\mu-\lambda|$ for some constant c whenever $|\mu-\lambda| < \delta$. Therefore

$$\|R(\lambda,\mu)\| < (c + \|J_\gamma^\eta\|)\varepsilon|\mu-\lambda|$$

if $|\mu-\lambda| < \delta$. Therefore

$$\|x_\eta(\mu) - x_\eta(\lambda) - A_\eta(\lambda)(\mu-\lambda)\| < (1-\kappa)^{-1}(c + \|J_\gamma^n\|)\varepsilon|\mu-\lambda|$$

if $|\mu-\lambda| < \delta$.

4. $A_\eta(\lambda)$ is a continuous function of λ. Let $\gamma > \eta$. $A_\eta(\lambda)$ is the fixed point of

$$G(A,\lambda) := D_1F_\eta^\eta(x_\eta(\lambda),\lambda)A + D_2F_\gamma^\eta(x_\gamma(\lambda),\lambda).$$

Therefore $A_\eta(\lambda)$ is continuous if for each $A \in A_\eta(\Lambda)$, the map $\mu \to G(A, \mu)$ is continuous. (The argument is similar to step 1.) Since $A_\eta(\lambda) = J_\gamma^\eta A_\gamma(\lambda)$, we have

$$\begin{aligned} G(A_\eta(\lambda), \mu) &= D_1 F_\eta^\eta(x_\eta(\mu), \mu) J_\gamma^\eta A_\gamma(\lambda) + D_2 F_\gamma^\eta(x_\gamma(\mu), \mu) \\ &= D_1 F_\gamma^\eta(x_\gamma(\mu), \mu) A_\gamma(\lambda) + D_2 F_\gamma^\eta(x_\gamma(\mu), \mu), \end{aligned}$$

which is a continuous function of μ because $\gamma > \eta$ and $x_\gamma(\mu)$ is continuous.

This completes the proof of the theorem in the case $r = 1$.

Suppose the theorem has been proved for some integer $r \geq 1$. Suppose moreover that we have shown:

(A) $D^r x_\eta(\lambda) \in L^r(\Lambda, X_\eta)$ satisfies an equation of the form

$$A = D_1 F_\eta^\eta(x_\eta(\lambda), \lambda))A + H_\gamma^\eta(\lambda),$$

where $\gamma > \eta$ and $H_\gamma^\eta(\lambda) \in L^r(\Lambda, X_\eta)$ has the following form. Let $\underline{i} = (i_1, \ldots, i_j)$ denote a j-tuple of integers. Then

$$H_\gamma^\eta(\lambda) = \sum a_{\underline{i}} D_1^j D_2^{r-|\underline{i}|} F_\gamma^\eta(x_\gamma(\lambda), \lambda) D^{i_1} x_\gamma \cdots D^{i_j} x_\gamma.$$

The sum is over all j-triples $\underline{i}$ with $0 \leq j \leq r$ and $0 \leq |\underline{i}| \leq r$, *except* the 1-triple $\underline{i} = (r)$. The possibility $\underline{i} = \varnothing$ is allowed; in that case the corresponding term is $D_2^r F_\gamma^\eta(x_\gamma(\lambda), \lambda)$. The coefficients $a_{\underline{i}}$ are not important for our purposes.

(B) $H_\xi^\eta = J_\gamma^\eta H_\xi^\gamma$ whenever $\xi > \gamma > \eta$.

(C) H_γ^η is independent of γ.

We shall show that the theorem is true for the integer $r+1$, and that (A), (B) and (C) hold with r replaced by $r+1$. Thus the theorem is true by induction. (Note that (A), (B), and (C) hold for $r = 1$ by step 2.).

Assume F_γ^η is C^{r+1} for $\gamma > \eta$, the theorem is true for the integer $r \geq 1$, and (A), (B) and (C) hold. Define $Y_\gamma := L^r(\Lambda, X_\gamma)$; for $\gamma > \eta$ define $\tilde{J}_\gamma^\eta \colon Y_\gamma \to Y_\eta$ by $\tilde{J}_\gamma^\eta A := J_\gamma^\eta \circ A$; for $\gamma > \eta$ define $\tilde{F}_\gamma^\eta \colon Y_\gamma \times \Lambda \to Y_\eta$ by

$$\tilde{F}_\gamma^\eta(A, \lambda) := D_1 F_\gamma^\eta(x_\gamma(\lambda), \lambda)A + H_\gamma^\eta(\lambda).$$

Define $\tilde{F}_\eta^\eta \colon Y_\eta \times \Lambda \to Y_\eta$ by

$$\tilde{F}_\eta^\eta(A, \lambda) := D_1 F_\eta^\eta(x_\eta(\lambda), \lambda)A + H_\gamma^\eta(\lambda)$$

for any $\gamma > \eta$. $\tilde{F}_\eta^\eta$ is well defined by (B).

Clearly $\{Y_\gamma\}$ with the maps $\{\tilde{J}_\gamma^\eta\}$ is a scale of Banach spaces. We claim that $\{\tilde{F}_\gamma^\eta\}$ is a scale invariant family of mappings of $\{Y_\gamma\}$ that satisfies the conditions of the theorem with $r = 1$. To show scale invariance, we note:

$$\begin{aligned} \tilde{J}_\gamma^\eta \tilde{F}_\xi^\gamma(A, \lambda) &= J_\gamma^\eta D_1 F_\xi^\gamma(x_\xi(\lambda), \lambda)A + J_\gamma^\eta H_\xi^\gamma(\lambda) \\ &= D_1 F_\xi^\eta(x_\xi(\lambda), \lambda)A + H_\xi^\eta(\lambda) = \tilde{F}_\xi^\eta(A, \lambda). \\ \tilde{F}_\gamma^\eta(\tilde{J}_\xi^\gamma A, \lambda) &= D_1 F_\gamma^\eta(x_\gamma(\lambda), \lambda) J_\xi^\gamma A + H_\gamma^\eta(\lambda) \\ &= D_1 F_\xi^\eta(x_\xi(\lambda), \lambda)A + H_\xi^\eta(\lambda) = \tilde{F}_\xi^\eta(A, \lambda). \end{aligned}$$

We have used (B) in the first computation and (C) in the second.

The "closed convex subset" of the theorem is Y_γ. Assumptions (1) and (2) of the theorem are clear: note that $\tilde{F}_\gamma^\eta$ is linear in A. Since F_γ^η is C^{r+1} for $\gamma > \eta$, each partial derivative of F_γ^η that appears in the definition of H_γ^η is at least C^1. Also, $x_\gamma(\lambda)$ is C^r by the induction hypothesis, and only derivatives of $x_\gamma(\lambda)$ through order $r-1$ appear in the definition of H_γ^η. Therefore H_γ^η is C^1 for $\gamma > \eta$. It follows from the case $r = 1$ of the theorem that the unique fixed point $A_\eta(\lambda)$ of $\tilde{F}_\eta^\eta$ is C^1; since $A_\eta(\lambda) = D^r x_\eta(\lambda)$, $x_\eta(\lambda)$ is C^{r+1}. The last line of the theorem implies that $D^{r+1}x_\eta(\lambda)$ satisfies the equation.

$$\begin{aligned} B &= D_1\tilde{F}_\eta^\eta(A_\eta(\lambda), \lambda)B + D_2\tilde{F}_\gamma^\eta(A_\gamma(\lambda), \lambda) \\ &= D_1 F_\eta^\eta(x_\eta(\lambda), \lambda)B + K_\gamma^\eta(\lambda), \end{aligned}$$

where

$$\begin{aligned} K_\gamma^\eta(\lambda) = D_1^2 F_\gamma^\eta(x_\gamma(\lambda), \lambda) D x_\gamma D^r x_\gamma \\ + D_1 D_2 F_\gamma^\eta(x_\gamma(\lambda), \lambda) D^r x_\gamma + D H_\gamma^\eta(\lambda). \end{aligned}$$

The arguments of step 2, applied to $\tilde{F}$, show that K_γ^η satisfies (B) and (C). From the formula for K_γ^η is also satisfies (A). □

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